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NON-BROKEN CIRCUITS OF REFLECTION GROUPS AND FACTORIZATION IN D_n

BY

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ABSTRACT

The set of non-broken circuits of a reflection group W, denoted NBC(W), appears as a basis of the Orlik–Solomon algebra of W. The factorization of their enumerating polynomial $\sum_{S\in \mathrm{NBC}(W)} t^{|S|} = \prod_{i=1}^{k}(1 + (d_i - 1)t)$ with respect to their cardinality involves the exponents $d_i - 1$ of W. A simple explanation of this factorization is known only for the symmetric groups S_n (Whitney [13]) and for the hyperoctahedral groups B_n (Lehrer [7]). In this paper, we present an elementary proof of the fact that the set NBC(W) of any reflection group W is in bijection with the group elements of W. We give a simple characterization of the non-broken circuits of the Weyl groups of type D_n and we use this characterization to prove the factorization of their enumerating polynomial.

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1. Introduction

Let W be a finite reflection group and let the subset Π_W of the real Euclidian space R^n be selected so that the set $R_W = \{\omega_v\colon v \in \Pi_W\}$ gives all the reflections of W. We shall refer to the elements of Π_W as the **roots** of W. Each element ω in W has several expressions of the form

$$\omega = \omega_{v_1}\omega_{v_2}\cdots\omega_{v_k}, \qquad v_i \in \Pi_W. \tag{1.1}$$

The smallest value of k in any decomposition given by equation (1.1) is called the **absolute length** of ω and denoted $\mathrm{al}(\omega)$ and the corresponding expression is said to be **totally reduced**. Let $\mathrm{Fix}(\omega) = \{v\colon v \in R^n \text{ and } \omega(v) = v\}$ be the fixed point space of ω. We recall the following results (see Carter [5])

PROPOSITION 1.1: *Let $v_1, v_2, \ldots, v_k \in \Pi_W$, then the decomposition $\omega = \omega_{v_1}\omega_{v_2}\cdots\omega_{v_k}$ is totally reduced if and only if the vectors $v_1, v_2, \ldots, v_k$ are linearly independent.*

PROPOSITION 1.2: *If a factorization $\omega = \omega_{v_1}\omega_{v_2}\cdots\omega_{v_k}$ is totally reduced, then $\{v_1, v_2, \ldots, v_k\}$ is a basis for the orthogonal complement of the fixed point space $\mathrm{Fix}(\omega)$ of ω and $n-k$ is the number of eigenvalues of ω equal to 1.*

We shall be concerned here with the polynomial

$$F_W(t) = \sum_{\omega \in W} t^{al(\omega)} \tag{1.2}$$

which we refer to as the **absolute length generator of** W. It was shown by Shephard–Todd ([10]) in a case by case analysis that

$$\sum_{i=0}^{n} a_i t^i = \prod_{i=1}^{n}(1 + (d_i - 1)t) \tag{1.3}$$

where a_k is the number of elements of W whose fixed point space in $\mathbb{R}^n$ has dimension $n-k$ and $d_1 - 1, d_2 - 1, \ldots, d_n - 1$ are the so called exponents of W. The integers d_i also give the degrees of any set of basic invariants of W. In view of Proposition 1.2, formula (1.3) is equivalent to

$$F_W(t) = \prod_{i=1}^{n}(1 + (d_i - 1)t). \tag{1.4}$$

A uniform proof of (1.3) was later given by Solomon [11]. Recall ([9]) that the characteristic polynomial $\chi_L(t)$ of a geometric lattice L is given by

$$\chi_L(t) = \sum_{x \in L} t^{r(L)-r(x)} \mu(\hat{0}, x) \tag{1.5}$$

where $\mu(\hat{0}, x)$ is the Möbius function and $r(x)$ is the rank of $x \in L$. Now let L_W be the lattice of intersections of the reflecting hyperplanes, associated to the set of roots Π_W, ordered by reverse inclusion. L_W is a geometric lattice whose atoms are indexed by the elements of Π_W. Let M be the complement of the union of the corresponding complexified reflecting hyperplanes. In [8], Orlik and Solomon show that the Poincaré polynomial $P_M(t)$ of M is given by

$$P_M(t) = \sum_{x \in L_W} \mu(\hat{0}, x)(-t)^{r(x)}. \tag{1.6}$$

Then using a result of Brieskorn ([4]) stating that $P_M(t) = \prod(1 + (d_i - 1)t)$, they obtain

$$(-t)^{r(L_W)} \chi_{L_W}(-1/t) = \sum_{x \in L_W} \mu(\hat{0}, x)(-t)^{r(x)} = \prod_{i=1}^{s}(1 + (d_i - 1)t). \tag{1.7}$$

Propositions 1.1 and 1.2 suggest a possible explanation for the equality of χ_{L_W} and $F_W(t)$. Recall that the Whitney–Rota theorem ([9]) states that for any geometric lattice L, the value $(-1)^{r(x)}\mu(\hat{0}, x) = |\mu(\hat{0}, x)|$ is equal to the number of NBC bases of x. Using this fact, the expression in (1.7) can be rewritten as

$$(-t)^{r(L_W)} \chi_L(-1/t) = \sum_{x \in L_W} \mathrm{card}(\mathrm{NBC}(x)) t^{r(x)} = \sum_{s \in \mathrm{NBC}(W)} t^{|s|} \tag{1.8}$$

where NBC(W) (respectively NBC(x)) is the set of non-broken circuits of L_W (respectively x).

The remarkable similarity between (1.2) and (1.8) strongly suggests finding a connection between group elements and NBC sets which explains the equality between these two polynomials when $L = L_W$. In fact such a connection does exist: given an NBC subset $S = \{v_1 < v_2 < \cdots < v_k\}$, let $\omega_S = \omega_{v_1}\omega_{v_2}\cdots\omega_{v_k}$. It develops that as S varies among all NBC subsets, ω_S describes all of the elements of W each once and only once:

THEOREM 3.1 (with A. Garsia): *Let $v_1\, v_2\, \ldots\, v_k$, be a set of roots of a reflection group W written in increasing order according to the total order $<_W$ on the atoms of Π_W. Let $\omega_{v_i}, \ldots, \omega_{v_k}$ be their corresponding reflections. The map Φ:* NBC$(W) \mapsto W$ *defined by $\Phi(v_1, \ldots, v_k) = \omega_{v_1}\omega_{v_2}\cdots\omega_{v_k}$ is a bijection from the NBC sets of W onto the group elements of W. Moreover* $\mathrm{al}(\Phi(S)) = r(S)$.

In this paper, we shall exploit this connection to give a combinatorial explanation for the common factorization of the polynomials $F_W(t)$ and $(-t)^{r(L_W)}\chi_{L_W}(-1/t)$ when W is the Coxeter group of type D_n. In Section 2, we give the basic definitions and recall how the factorization of the enumerating polynomial NBC(t, W) of non-broken circuits, defined by equation (1.8), is done combinatorially for the symmetric group S_n and the hyperoctahedral group B_n. In Section 3, we establish the bijection between NBC sets and the group elements of any reflection group W and we use it to give a first proof of the factorization of the polynomial NBC(t, D_n). In Section 4, we characterize the circuits and the non-broken circuits of D_n and in Section 5 we give a second proof of the factorization of NBC(t, D_n) using only combinatorial properties of the NBC sets of D_n.

2. Preliminaries

Let W be a finite reflection group and let Π_W be its corresponding system of roots which is in one-to-one correspondance with the set of reflections of W. The roots for the groups of type A_{n-1}, B_n and D_n are represented respectively by the sets

$$\begin{aligned} \Pi_{A_{n-1}} &= \{e_i - e_j\}_{1\leq i<j\leq n}, \\ \Pi_{B_n} &= \{e_i \pm e_j\}_{1\leq i<j\leq n} \cup \{e_i\}_{i=1\ldots n}, \\ \Pi_{D_n} &= \{e_i \pm e_j\}_{1\leq i<j\leq n}, \end{aligned} \tag{2.1}$$

where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis for R^n.

2.1 For any subset $S \subseteq \Pi_W$ we define the **closure** $\overline{S}$ of S as follows:

$$\overline{S} = \Pi_W \cap L(S) \tag{2.2}$$

where $L(S)$ is the linear span of S in R^n. Let C_W be the collection of closed subsets of Π_W. When we order the elements of C_W by inclusion, we obtain a ranked poset $(C_W, \subseteq)$ that becomes a geometric lattice if one defines the meet and join operations to be $x \wedge y = x \cap y$ and $x \vee y = \overline{x \cup y}$ (see Birkhoff ([3])). The elements of rank one in the lattice $(C_W, \subseteq)$, called the **atoms**, are the vectors of Π_W. It is a well known fact that the map: $\bar{S} \mapsto \bar{S}^{\perp}$ gives a lattice isomorphism between C_W and the lattice of intersections of hyperplanes L_W.

Let $S = \{v_1, v_2, \ldots, v_p\} \subseteq C_W$ be a set of atoms of $(C_W, \subseteq)$ and let $\vee S = v_1 \vee v_2 \vee \cdots \vee v_p$, then the rank $r(\vee S)$ of $\vee S$ is given by $r(\vee S) = \dim(L(S))$ and we have $r(\vee S) \leq |S|$.

2.2 A set $S = \{v_1, v_2, ..., v_p\} \subseteq \Pi_W$ is said to be **independent** if and only if $r(\vee S) = |S|$. Otherwise S is **dependent**.

2.3 $S = \{v_1, v_2, \ldots, v_p\}$ is said to be a **base** for an element x of the lattice $(C, \subseteq)$ if and only if S is independent and $\vee S = x$.

2.4 A **circuit** is a dependent set $S \subseteq \Pi_W$ such that all its proper subsets $T \subset S$ are independent.

2.5 Given a total order $<_W$ on the set Π_W of atoms, we say that $S = \{v_1, v_2, \ldots, v_p\}$ is a **broken circuit**, denoted BC, if there is an element $v \in \Pi_W$ such that $v <_W v_k$ for all $k = 1, \ldots, p$ and $S \cup \{v\}$ is a circuit. In other words, the broken circuits are obtained from the circuits by removing the smallest atom.

2.6 A **non-broken circuit**, denoted NBC, is a set of atoms that does not contain any broken circuit. It can be shown that any NBC is independent.

2.7 The **enumerating polynomial** of the set of non-broken circuits of W with respect to the rank is defined as follows

$$\mathrm{NBC}(t, W) = \sum_{S \in \mathrm{NBC}(W)} t^{r(S)}. \tag{2.3}$$

We are interested in the combinatorial factorization of $\mathrm{NBC}(t, W)$. We first start with the symmetric group.

THE GROUP S_n. To any decomposition

$$\sigma = \tau_1 \tau_2 \cdots \tau_k \tag{2.4}$$

of a permutation $\sigma \in S_n$ as a product of transpositions, we associate a graph $G = ([n], R)$ whose vertices are in the set $[n] = \{1, 2, \ldots, n\}$ and with set of edges $R = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$ when the transpositions are $\tau_1 = (i_1, j_1), \ldots, \tau_k = (i_k, j_k)$.

Note that since there is a one-to-one correspondance between the atoms of $(C_{S_n}, \subseteq)$ and the transpositions of S_n, we shall speak indifferently about atoms or transpositions of S_n.

PROPOSITION 2.1: *A set of atoms (transpositions) of S_n is a circuit if and only if its associated graph is a cycle.*

Proof: This is a consequence of Proposition 1.1 and the fact (see Dénes [6]) that a decomposition as in 2.4 is minimal if and only if the associated graph is a tree. ∎

The total order we give to the set of atoms (j,k), $j > k$, of S_n is the lexicographic order on the pairs (j,k):

$$(2.5) \qquad (j,k) <_L (l,m) \iff \begin{cases} j < l & \text{or} \\ j = l & \text{and} \quad k < m. \end{cases}$$

PROPOSITION 2.2 (see [1]): *A set of atoms of S_n is NBC if and only if it does not contain a pair of atoms of the form $\{(j,l),(j,k)\}$ with $j > l$ and $j > k$.*

After Garsia, we call such a pair of atoms a **camel hump**. We represent a set of atoms of the form $\{(m,1),(m,2),\ldots,(m,m-1)\}$ by the graph in Figure 1(a) and we refer to it as an m-**hand** of S_n and denote it by $H_m(S_n)$.

The NBC sets of S_n are obtained by selecting at most one atom from each m-hand $H_2(S_n)$, $H_3(S_n)$, $\ldots, H_n(S_n)$. The decomposition of the polynomial $\mathrm{NBC}(t, S_n)$

$$(2.6) \qquad \sum_{S \in \mathrm{NBC}(S_n)} t^{r(S)} = \prod_{i=1}^{n-1} (1 + it)$$

is now a direct consequence of Proposition 2.2 when we give the weight t to the atoms in the hands of S_n.

(a) m-hand of S_n (b) m-hand of B_n

Figure 1. Hands of S_n and B_n.

THE GROUP B_n. From equation (2.1), we see that we have three types of reflections in the hyperoctahedral group B_n. We denote them by (i,j), $(\bar{i},\bar{j})$, $(i,\bar{i})$ when they respectively correspond to the reflections in the hyperplanes orthogonal to the vectors $e_i - e_j$, $e_i + e_j$ and e_i.

Similarly to S_n, we associate to each decomposition (1.1) of $\omega \in B_n$, and consequently to each NBC set of B_n, a graph G with set of vertices $\{1,\ldots,n\}$ and set of edges given by the set of reflections $\{v_1,\ldots,v_k\}$. A reflection (or atom) of type (i,j) will be represented in G by a straight line between the vertices i and j. A reflection of type $(\bar{i},\bar{j})$ will be represented by a curved line between i and j and a reflection $(i,\bar{i})$ by a loop on the vertex i.

We define a camel hump of B_n to be a pair of atoms of one of the forms $\{(j,l),(j,k)\}$, $\{(j,l),(\bar{j},\bar{k})\}$, $\{(\bar{j},\bar{l}),(\bar{j},\bar{k})\}$ with $j > l$ and $j > k$. The camel humps are represented in Figure 2 where $l = k$ is possible for the second pair. These camel humps are broken circuits and any broken circuit of B_n contains a camel hump. An m-hand of B_n is defined to be the set

$$H_m(B_n) = \{(m,1),(m,2),\ldots,(m,m-1),(\bar{m},\bar{1}),\ldots,(\bar{m},\overline{m-1}),(m,\bar{m})\}$$

graphically represented in Figure 1(b). We define the total order on the atoms of B_n to be the one that is obtained by reading from left to right in Figure 3 the atoms in the m-hands of B_n.

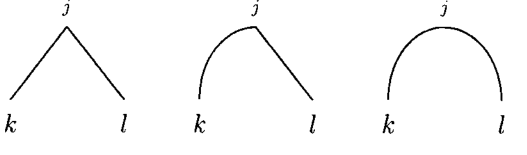


Figure 2. Camel humps of B_n.

PROPOSITION 2.3 (see Lehrer [7]): *The NBC sets of B_n are obtained by selecting at most one atom from each m-hand $H_1(B_n)$, $H_2(B_n)$, ..., $H_n(B_n)$.*

Again, we use Proposition 2.3 to factor the polynomial NBC(t,B_n) by giving the weight t to each atom in the hands $H_1(B_n)$, $H_2(B_n),\ldots,H_n(B_n)$ shown in Figure 3 and we obtain:

$$\sum_{S\in \mathrm{NBC}(B_n)} t^{r(S)} = \prod_{i=1}^{n}(1+(2i-1)t). \tag{2.7}$$

3. A bijection between NBC sets and the group elements of W

We first recall (see [5]) that for any decomposition (1.1) of $\omega \in W$, the set of vectors $\{v_1, v_2, \ldots, v_k\}$ spans the orthogonal complement $\mathrm{Fix}(\omega)^\perp$ of the fixed point space of ω.

THEOREM 3.1 (with A. Garsia): *Let $v_1\ v_2 \ldots v_k$ be a set of roots of a reflection group W written in increasing order according to the total order $<_W$ on the atoms of Π_W. Let $\omega_{v_i}, \ldots, \omega_{v_k}$ be their corresponding reflections. The map Φ: $\mathrm{NBC}(W) \mapsto W$ defined by $\Phi(v_1, \ldots, v_k) = \omega_{v_1}\omega_{v_2}\cdots\omega_{v_k}$ is a bijection from the NBC sets of W onto the group elements of W. Moreover $\mathrm{al}(\Phi(S)) = r(S)$.*

Proof: Let $\{v_1, v_2, \ldots, v_k\}$ and $\{u_1, u_2, \ldots, u_r\}$ be two NBC sets of W. Suppose $\omega_{v_1}\omega_{v_2}\cdots\omega_{v_k} = \omega_{u_1}\omega_{u_2}\cdots\omega_{u_r}$ and $v_1 <_W u_1$, then the set $\{v_1, u_1, \ldots, u_r\}$ is linearly dependent since the sets $\{v_1, \ldots, v_k\}$ and $\{u_1, \ldots, u_r\}$ generate the same space $\mathrm{Fix}(\omega)^\perp$. This implies that $\{v_1, u_1, \ldots, u_r\}$ contains a circuit of the form $\{v_1, u_{j_1}, \ldots, u_{j_l}\}$ with $l \leq r$ so that $\{u_{j_1}, \ldots, u_{j_l}\}$ is a broken circuit. But this contradicts the hypothesis that $u_1, u_2, \ldots, u_r$ is NBC. So we must have $v_1 \geq u_1$ and by symmetry $v_1 = u_1$. Thus we have

$$\omega_{v_2}\omega_{v_3}\cdots\omega_{v_k} = \omega_{u_2}\omega_{u_3}\cdots\omega_{u_r}. \tag{3.1}$$

If we repeat the same argument on 3.1, we obtain $v_2 = u_2, \ldots, v_k = u_k$ and $k = r$ and the map Φ is one-to-one.

Next we want to show that for each element $\omega \in W$, there exists a minimal decomposition $\omega = r_{u_1}r_{u_2}\cdots r_{u_k}$ of ω as a product of reflections such that $u_1 <_W u_2 <_W \cdots <_W u_k$ and the set $\{u_1 \ldots u_k\}$ is an NBC set. We proceed by induction on $\mathrm{al}(\omega)$. Let $\mathrm{al}(\omega) = k$ and $\omega = \omega_{v_1}\omega_{v_2}\cdots\omega_{v_k}$ be any minimal decomposition. Choose u_1 in the linear span of $S = \{v_1, v_2, \ldots, v_k\}$ so that r_{u_1} is the smallest reflection with respect to the total order $<_W$. We have that $\mathrm{al}(r_{u_1}\omega) < k$ because $\det(r_{u_1}\omega) \neq \det(\omega)$ implies that $\mathrm{al}(r_{u_1}\omega) \neq k$ and $\mathrm{al}(r_{u_1}\omega) > k$ would imply that $S \cup \{u_1\}$ is linearly independent, which is impossible. Thus let $r_{u_1}\omega = \omega_{s_1}\omega_{s_2}\cdots\omega_{s_h}$ be a minimal decomposition with $h < k$. We have that $\omega = r_{u_1}\omega_{s_1}\omega_{s_2}\cdots\omega_{s_h}$ is a minimal decomposition with $r_{u_1} < \omega_{s_i}$ for all $i = 1, \ldots, h$ (so that $h = k - 1$). Now by induction hypothesis $\omega' = \omega_{s_1}\omega_{s_2}\cdots\omega_{s_h}$ can be written as a product of reflections $r_{u_2}\cdots r_{u_k}$ that corresponds to an NBC set $\{u_2, \ldots, u_k\}$. The product $r_{u_1}r_{u_2}\cdots r_{u_k}$ is easily seen to be also an NBC set of W and the proof is complete. ∎

THE GROUP D_n. We give our first proof of the identity

$$\sum_{S\in \mathrm{NBC}(D_n)} t^{r(S)} = (1+(n-1)t)\prod_{i=1}^{n-1}(1+(2i-1)t) \tag{3.2}$$

by using the factorization of $\mathrm{NBC}(t, B_n)$ given in (2.7). In view of Theorem 3.1, identity (3.2) is equivalent to

$$\sum_{\omega\in D_n} t^{\mathrm{al}(\omega)} = (1+(n-1)t)\prod_{i=1}^{n-1}(1+(2i-1)t). \tag{3.3}$$

The graphs representing the m-hands of B_n are given in Figure 3. Consider the NBC sets of B_n obtained by choosing at most one atom in each one of the first $n-1$ hands and then choosing at most one atom in the last hand so that the product of the corresponding reflections is in D_n. The choice of the last atom depends on the parity of the number of atoms of type $(i,\bar{i})$ already chosen. The bijection Φ in Theorem 3.1 that relates the NBC sets of B_n with its elements, when restricted to these NBC sets, gives a bijection with the set of elements of D_n.

Figure 3. m-Hands of B_n.

We now proceed to the proof of identity (3.3) by induction on n. Since D_3 is isomorphic to S_4, we have from (2.6)

$$\sum_{\omega\in D_3} t^{\mathrm{al}(\omega)} = (1+t)(1+2t)(1+3t). \tag{3.4}$$

When we proceed to the choice of at most one atom in each of the first $n-1$ hands of B_n, if we obtain an element of D_{n-1}, then we have the choice among $2(n-1)$ atoms in the last hand $H_n(B_n)$ in order to obtain an element of D_n. The absolute length generator for these elements of D_n is by induction hypothesis the product of the absolute length generator of D_{n-1} with $(1+2(n-1)t)$ which gives:

$$(1+t)(1+3t)\cdots(1+(2n-5)t)(1+(n-2)t)[1+2(n-1)t]. \tag{3.5}$$

Similarly, the choice of atoms in the first $n-1$ hands that gives an element of $B_{n-1} \setminus D_{n-1}$ forces the choice of the atom $(\overline{n}, \overline{n})$ in the last hand to get an element of D_n. And since $1+(2n-3)t = (n-1)t + (1+(n-2)t)$, the absolute length generator for these elements of D_n is obtained by multiplying the absolute length generator of $B_{n-1} \setminus D_{n-1}$ by t which is

$$(1+t)(1+3t)\cdots(1+(2n-5)t)(n-1)t^2. \tag{3.6}$$

The sum of the enumerating polynomials (3.5) and (3.6) gives

$$(1+t)(1+3t)\cdots(1+(2n-3)t)(1+(n-1)t) \tag{3.7}$$

which is the required result. ∎

4. The non-broken circuit sets of D_n

In this section we give a description of the NBC sets of atoms of D_n. Recall that the atoms are the vectors of Π_{D_n}. To describe the NBC sets we associate to D_n a graph G_{D_n}, made of **positive** and **negative** edges. The graph G_{D_n} is similar to the one described for B_n, the only difference being that in G_{D_n} there are no loops. The positive edges (represented by straight lines) correspond to the roots of the form $(e_i - e_j)$, while the negative edges (represented by curved lines) correspond to the roots of the form $(e_i + e_j)$.

As we saw in the preliminaries, a circuit is a minimally dependent set $S \subseteq \Pi_{D_n}$. Thus the circuits of Π_{D_n} will be described by looking at their corresponding set of edges in G_{D_n}. Formally the atoms, the circuits and the NBC sets are all subsets of Π_{D_n} but when there is no risk of confusion we shall identify the atoms of Π_{D_n} with the reflections of D_n or with the edges of G_{D_n}.

In [14], Zaslavsky introduced the notions of signed graph and of switching. His theorem 5.1 contains, among other things, the description without proofs of the circuits corresponding to signed graphs. Our graph G_{D_n} is a special case of a signed graph and we include elementary proofs (Theorem 4.1 to Theorem 4.3) for the classification of the configurations that correspond to circuits.

THEOREM 4.1: *A cycle C in G_{D_n} corresponds to a circuit of Π_{D_n} if and only if the number of negative edges of C is even.*

Proof: By a cycle we mean a sequence of k edges:

$$C = i_1 \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_k \longrightarrow i_1$$

between a set $\{i_1, i_2, \ldots, i_k\}$ of k distinct vertices. As we pass through an edge $(i\ j)$, (or $(\bar{i}, \bar{j})$) going through i first and then through j, we consider the corresponding vector to be $e_i - e_j$ (or $e_i + e_j$) even if $i > j$. (Clearly there is no loss of generality by doing so.)

A. DEPENDENCY: We first show that if the number h of negative edges in a cycle C is even then the corresponding set of vectors is a dependent set of Π_{D_n}.
(a) Let $h = 0$. Since there is no negative edges, the corresponding vectors are all of the form $e_{i_j} - e_{i_{j+1}}$. Thus the telescoping sum:

$$\sum_{s=1}^{k} (e_{i_s} - e_{i_{s+1}}) \tag{4.1}$$

where $e_{i_{k+1}} = e_{i_1}$ is clearly equal to $(0, \ldots, 0)$.

(b) Let $h = 2, 4, 6 \ldots$. In that case the coefficients can be chosen as follows: pick a negative edge in the cycle, and let the coefficient of the corresponding vector be $(+1)$. Now going clockwise through C, let the coefficients of the vectors corresponding to the subsequent edges, all the way to the next negative edge (including that one), be equal to (-1). If we have traversed all the edges of the cycle we are done; if not then the coefficients of the vectors corresponding to the edges following that last one will be $(+1)$ all the way to the next negative edge and so on , alternating between $(+1)$ and (-1) until we have gone through the entire cycle C. What determines the change of sign is when one traverses a negative edge, then the coefficients of the vectors corresponding to the subsequent edges will have a sign opposite to the sign that was attributed to this negative edge. This procedure is illustrated in Figure 4. The **plus** (respectively **minus**) sign above an edge means that the corresponding coefficient is $+1$ (respectively -1).

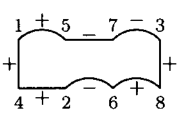


Figure 4. A cycle with an even number of negative edges.

To see that this linear combination gives $(0, \ldots, 0)$, one need only notice the following fact. If $(e_{j_1} + e_{j_2})$ and $(e_{j_m} + e_{j_{m+1}})$ are two negative edges of C with only positive edges between them, then

$$(e_{j_1} + e_{j_2}) - \Big(\sum_{k=2}^{m-1} e_{j_k} - e_{j_{k+1}}\Big) - (e_{j_m} + e_{j_{m+1}}) \ = \sum_{k=1}^{m} (e_{j_k} - e_{j_{k+1}}) \ . \tag{4.2}$$

Indeed a simple calculation shows that each side of equation (4.1), is equal to $e_{j_1} - e_{j_{m+1}}$. Observe that the right hand side of (4.2) is composed of vectors corresponding to positive edges only. Thus using equation (4.2) for every pair of negative edges in the cycle yields a telescopic sum of the form (4.1) which completes the proof of A.

B. MINIMALITY: To show that this set of vectors is a circuit we only need to prove the minimality of the set. That is, if we remove one edge of the cycle the new corresponding set of vectors is linearly independent. Let $(e_{i_j} \pm e_{i_{j+1}})$ be the vector corresponding to the removed edge. Assuming that the set of vectors $\{(e_{i_1} \pm e_{i_2}), \ldots, (e_{i_{j-1}} \pm e_{i_j}), (e_{i_{j+1}} \pm e_{i_{j+2}}), \ldots, (e_{i_k} \pm e_{i_1})\}$ is linearly dependent, implies that there exists coefficients $c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_k$ not all equal to 0 such that

$$\sum_{\substack{s=1 \\ s\neq j}}^{k} c_s(e_{i_s} \pm e_{i_{s+1}}) = (0, \ldots, 0)$$

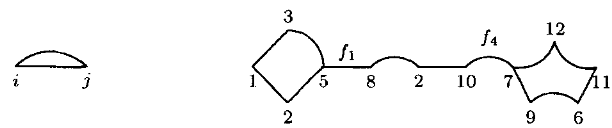
where $e_{i_{k+1}} = e_{i_1}$. The only way to obtain a zero in the i_j th and i_{j+1} th coordinates of the vector $(0, \ldots, 0)$ is when $c_j = c_{j+1} = 0$. But this recursively forces the coefficients $c_{j-1}, \ldots, c_1$ and $c_{j+2}, \ldots, c_k$ to be all zero, which contradicts our assumption.

C. CONVERSE: We now prove that if a cycle has an odd number of negative edges then the corresponding set of vectors is linearly independent. Suppose on the contrary that these vectors are linearly dependent. Once more, this means that there exists coefficients $c_1, c_2, \ldots, c_k$, not all equal to zero such that

$$c_1(e_{i_1} \pm e_{i_2}) + c_2(e_{i_2} \pm e_{i_3}) + \cdots + c_k(e_{i_k} \pm e_{i_1}) = (0, 0, \ldots, 0)\ .$$

This forces the equalities $c_1 = \pm c_k$; $c_k = \pm c_{k-1}$; $c_{k-1} = \pm c_{k-2}$; ...; $c_2 = \pm c_1$, which in turn yields $c_1 = \pm c_1$. Observe that a positive edge $(e_{i_j} - e_{i_{j+1}})$ gives rise to the equation: $c_{j+1} = c_j$, while a negative edge $e_{i_s} + e_{i_{s+1}}$ yields: $c_{s+1} = -c_s$. With this in mind it is easy to see that an odd number of negative edges yields $c_1 = -c_1$, thus, $c_1 = 0$. But, by a domino effect this forces all the c_i to be equal to zero, contradicting our assumption. ∎

The set $\{(e_i + e_j),\ (e_i - e_j)\}$ is not a circuit. Graphically, it corresponds to two edges (a positive and a negative one) between the vertices i and j. Call this configuration a **bubble** (see Figure 5(a)).



(a) bubble

(b) CC-configuration

Figure 5. Bubble and CC configuration.

The bubble can be thought of as a cycle having one negative edge. This terminology agrees with the fact that the set of vectors corresponding to a cycle having an odd number of negative edges is not a circuit. Therefore, from now on the word cycle shall include this degenerate case . We now describe a new configuration of edges in G_{D_n} that leads to a circuit in Π_{D_n} .

Let CC be the configuration consisting of: two cycles C_1, C_2, each of them containing an odd number of negative edges, connected by a unique chain $F = \{f_1, f_2, \ldots, f_t\}$ $(t \geq 0)$, of positive or negative edges. The intersection between F and C_1 and C_2 consists of exactly two vertices when the chain is not empty. One is the vertex a_1, common to f_1 and C_1 while the second one is the vertex b_1, common to f_t and C_2. If the chain is empty then the intersection between C_1 and C_2 consists of a single vertex a_1. Figure 5(b) gives an example of a CC configuration where $a_1 = 5$, $b_1 = 7$ and $t = 4$.

THEOREM 4.2: *The set of vectors of* Π_{D_n} *corresponding to a* CC *configuration is a circuit.*

Proof: We first give an example in which there exist coefficients $c_1, c_2, \ldots$, not all equal to zero such that

$$\sum c_t(e_{i_t} \pm e_{i_{t+1}}) = (0, \ldots, 0).$$

Let the CC configuration be as in Figure 5(b). The following linear combination:

$$(e_5 - e_2) + (e_2 - e_1) + (e_1 - e_3) + (e_3 + e_5) - 2(e_5 - e_8) - 2(e_8 + e_2)$$
$$+2(e_2 - e_{10}) + 2(e_{10} + e_7) - (e_7 + e_{12}) + (e_{12} + e_{11}) - (e_{11} - e_6) - (e_6 + e_9) + (e_9 - e_7)$$

yields the zero vector. The idea of the proof is the following one: the coefficients of the vectors corresponding to the edges of the two cycles C_1, C_2 will be (± 1), while the coefficients of the edges of the chain F will be (± 2). The change of sign will still be determined by whether one traverses a negative edge or not. The only remaining point is to decide where to start. Pick the vertex a_1 and go through the cycle C_1 in a clockwise manner. The vector corresponding to this *first* edge of C_1 will be assigned the value $(+1)$. From there, the coefficients of the successive edges of C_1 will be (± 1) depending on whether or not one traverses a negative edge. Once we reach a_1 again, one traverses the chain F taking care this time to assign the value (± 2) to the corresponding coefficients, until one reaches the second cycle C_2. This cycle should also be traversed clockwise (from b_1) but the value of the coefficients associated to its edges should be set back to (± 1).

The fact that all the degrees of the vertices are equal to two, except for that of a_1 and b_1 which have degree three, or four if F is the empty chain, is the reason why the coefficients of the edges of the chain need to be twice the one of the cycles.

Note that in the degenerate case where the cycle is a bubble, the clockwise direction has no meaning. So set $(+1)$ for the negative edge of C_1 and consequently (-1) for the positive one. Thus the first edge of the chain F will be assigned the coefficient (-2) after which we continue as described above. In all cases this non-trivial linear combination is equal to zero.

To show the minimality, let us assume that if we take away any vector of CC there is a non trivial linear combination of the remaining vectors of CC that yields the zero vector. Note that taking away an edge creates at least one vertex of degree one. But, as we saw earlier this forces one of the coefficients of the given linear combination to be equal to zero.

If the removed edge belonged to C_1 (or C_2), all the coefficients of the vectors corresponding to the edges of that cycle C_1 (C_2) and of the chain F, will have to be equal to zero, by the domino argument of Theorem 4.1. Hence, one is left with a cycle C_2 (C_1) having an odd number of negative edges, which corresponds to a set of linearly independent vectors. Thus the only linear combination yielding the zero vector is the trivial one.

If the removed edge belongs to the chain F, this forces all the coefficients of the vectors of that chain to be zero. One is then left with the two disjoint cycles C_1 and C_2 and this completes our proof. ∎

We claim that the only configurations of edges of G_{D_n} that yield circuits are the cycles with an even number of edges or the CC configurations described above. Indeed, the way to obtain circuits is by adding a new vector v to a set S of vectors, linearly independent, in such a way that the new set of vectors $S \cup \{v\}$ becomes a minimally dependent set. We shall characterize the sets of vectors that are linearly independent.

The set of vectors corresponding to a tree in G_{D_n}, is a linearly independent set. If we add an edge so that a cycle is created then it is immediate to see that we get a circuit only if the tree has no branches (i.e. the tree was a chain) and if the number of negative edges in that cycle is even. Indeed a tree structure with such a cycle appended to one of its vertices is not a circuit, because the minimality requirement is not satisfied. A tree with a cycle having an odd number of negative edges appended to one of its vertices is a linearly independent set. Adding an edge so that a new cycle is created gives a circuit only if the new cycle has an odd number of edges and, if the only remaining edges, not belonging to the 2 cycles, form a chain between the two cycles. In other words, a CC configuration to which a tree-like structure would be appended is not a circuit because the minimality requirement is not satisfied.

Now consider a cycle with an odd number of negative edges. Adding an edge between two of the vertices of the cycle does not create a circuit for the following reasons. The added edge creates two cycles sharing this edge. One of these two cycles has an even number (possibly zero) of negative edges, which we know is a circuit, while the edges of the other cycle create a set of vectors which is not minimally dependent. A similar argument shows that if one has two (2) disjoint cycles, each with an odd number of negative edges, the only way to get a circuit is by adding an edge between the two cycles.

These arguments prove the following theorem.

THEOREM 4.3: *A set of vectors* $S = \{v_1, \ldots, v_k\} \subseteq \Pi_{D_n}$ *is a circuit iff it is a* CC *configuration or if it corresponds to a cycle with an even number of negative edges.*

We recall that the NBC sets of atoms are the sets of atoms which do not contain any broken circuits. A broken circuit is a set of atoms corresponding to a circuit from which we removed the smallest atom. From now on the total order for the atoms of D_n is the following one:

$$(2,1),\ (\bar{2},\bar{1}),\ (3,1),\ (3,2),\ (\bar{3},\bar{1}),\ (\bar{3},\bar{2}),\ (4,1),$$
$$(4,2),\ (4,3),\ (\bar{4},\bar{1}),\ (\bar{4},\bar{2}),\ (\bar{4},\bar{3}),\ \ldots,\ (\bar{n},\overline{n-1})$$

Note that we represent an atom $(e_i - e_j)$ $((e_i + e_j))$ by its corresponding reflection (j, i), $((\bar{j}, \bar{i}))$ where we insist (as for S_n) that $j > i$. Observe that to a set of atoms of D_n there corresponds a set of edges in the graph G_{D_n}. This set of edges forms a subgraph of G_{D_n}. It is in those terms that we characterize the NBC sets of D_n. Before doing so, we shall first describe the three different types of broken circuits. If a circuit corresponds to a cycle then removing the smallest edge yields a chain which is said to be a BC of type 1. If the circuit corresponds to a CC configuration then removing one edge yield two possible configurations; a connected component shown in Figure 6(a) (BC of type 2); or a disconnected component shown in Figure 6(b) (BC of type 3). The dotted lines represent edges that can be either positive or negative.

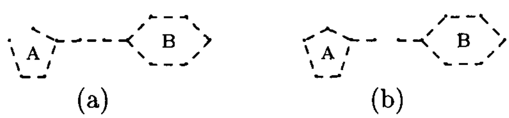


Figure 6. Broken circuits of type 2 and 3.

To ease the burden of notation we shall give a name to a few configurations. The camel hump configurations will refer to any pair of atoms of the form given in Figure 2 where $i \neq j$. The **chain-bubble** configuration is shown in Figure 7(a), where $i < j < i_3 < \cdots < i_k$ or $j < i < i_3 < \cdots < i_k$.

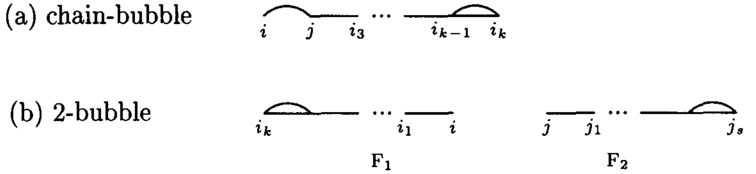


Figure 7. Chain-bubble and 2-bubble configurations.

Note that there are only two negative edges in this configuration: The one between the vertices i and j and the other one between the vertices i_{k-1} and i_k. Lastly the **2-bubble** configuration shall refer to the one given in Figure 7(b).

This configuration consists of two disjoint chains F_1 and F_2, each of them ending with a bubble and containing only positive edges, except for the bubble. They satisfy $i < i_1 < \cdots < i_k$, $j < j_1 < \cdots < j_s$ and the fact that if the smallest vertex i is in the connected component F_1 then, the next smallest vertex j, has to be in the connected component F_2.

THEOREM 4.4: *A set S of atoms of D_n is an* NBC *set if and only if the associated graph does not contain any subgraph of the form:*

(1) *camel hump,*
(2) *chain-bubble,*
(3) *2-bubble.*

Proof: The condition is clearly necessary. Indeed, a set S of atoms corresponding to any configurations of the form $1, 2$, or 3 is a broken circuit. In each case, the removed edge is $(i\ j)$ or $(-i\ \ -j)$, which we denote by $(\pm i\ \ \pm j)$. Thus we need only show that if S contains a broken circuit then it contains a broken circuit of the form $1, 2$, or 3. As we mentioned above the broken circuits are of three types.

CASE 1: Let the broken circuit correspond to the path

$$\underset{\rightarrow}{j}\underset{\rightarrow}{\pm i_1}\underset{\rightarrow}{\pm i_2}\pm\cdots\underset{\rightarrow}{\pm i_{k-1}}\underset{\rightarrow}{\pm i_k}\pm i$$

with $j > i$ and $(\pm i\ \ \pm j)$ the removed edge. Since this edge must be the smallest in the atom order we have adopted, we claim that i_k is greater than both i and j. Indeed, assume if possible that $i_k < i$. Then in the order described above $(\pm i, \pm i_k) < (\pm j, \pm i)$ which contradicts the assumed minimality of $(\pm j, \pm i)$, thus $i_k > i$. A similar argument yields that $i_k > j$. This given, let i_s be the maximum of $j = i_0, i_1, i_2, \ldots, i_k, i_{k+1} = i$. By our previous argument we can conclude that $0 < s < k+1$. Thus i_{s-1} and i_{s+1} are both smaller than i_s. But this implies that the original broken circuit contains a camel hump.

CASE 2: Let the broken circuit correspond to the configuration given in Figure 6(a). In the cycles A and B, a number of edges greater than 2 yields a broken circuit that contains a camel hump (see case 1 above). Thus, consider the case where the broken circuit BC is the set of atoms represented in Figure 8(a), where the smallest atom was (j, i). Remember that the dotted lines represent the edges that can be either positive or negative.

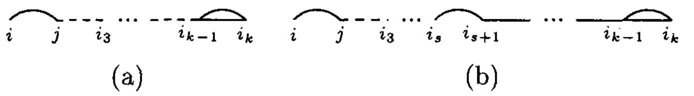


Figure 8. Broken circuits of type 2 for degenerate case.

We claim that BC contains a camel hump or a chain-bubble broken circuit.

Let $i < j$. Not having $j < i_3 < i_4 < \cdots < i_{k-1} < i_k$, implies that BC contains a camel hump. So let $i < j < i_3 < \cdots < i_{k-1} < i_k$, and consider the set of atoms represented in Figure 8(b), where $(-i_s \ - i_{s+1})$ is the last encountered negative edge in the path going from i to i_{k-1}. This is clearly a chain-bubble situation.

Let $j < i$. If $j > i_3$ then we have that $(\pm j, \pm i_3) < (i,j)$, but this contradicts the minimality of (i,j). A similar argument shows that $i < i_3$, thus $j < i < i_3$ and reasoning as above yields that the broken circuit contains a chain-bubble.

Case 3: Let the broken circuit correspond to the configuration shown in Figure 9,

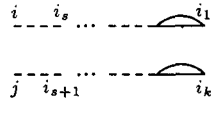


Figure 9. Broken circuit of type 3 for degenerate case.

Note that in Figure 9, if the bubbles were to be cycles of length ≥ 3, then the corresponding broken circuit would contain a camel hump. Hence it is sufficient to consider these configurations.

Without loss of generality let $i < j$. The minimality property of the edge $(\pm j, \pm i)$ forces $i_s > j > i$. Considering the situations where the broken circuit does not contain camel humps implies that $i < i_s < i_{s-1} < \cdots < i_1$. Now we either have $j < i_{s+1} < \cdots < i_k$ or $j > i_{s+1} > \cdots > i_t < i_{t+1}, \cdots, i_k$, for some t. The first case yields a broken circuit that contains a 2-bubbles broken circuit. The second case cannot occur since otherwise the minimality property of the edge $(\pm j, \pm i)$ would not be preserved.

Observe that in cases 2 and 3 the **chain-portion** of the configuration can be empty. This, per se, requires an argument of its own. But as it turns out the arguments concerning these extreme cases are simpler than the general one and we omit the details. ∎

5. Factorization

We now give a proof of the factorization of the polynomial $\mathrm{NBC}(t, D_n)$ using only properties of the NBC sets of D_n.

THEOREM 5.1: *We have*

$$\sum_{S \in \mathrm{NBC}(D_n)} t^{|S|} = (1+t)(1+3t)\cdots(1+(2n-3)t)(1+(n-1)t).$$

Proof (Induction on n): If we add an atom (n,i) or $(\bar{n},\bar{i})$ to an NBC set of D_{n-1}, we necessarily obtain an NBC set of D_n. Since there are $2(n-1)$ atoms of these two types, the enumerating polynomial $Q_n(t)$ of the NBC sets of D_n that contain at most one of the atoms (n,i), $(\bar{n},\bar{i})$ is by induction equal to

$$Q_n(t) = \mathrm{NBC}(t, D_{n-1})(1+2(n-1)t). \tag{5.1}$$

Next, we claim that the enumerating polynomial $P_n(t)$ of the NBC sets of D_n that contain a bubble $\{(n,i),(\bar{n},\bar{i})\}$, $1 \le i \le n-1$ is equal to

$$P_n(t) = (1+t)(1+3t)\cdots(1+(2n-5)t)t^2 \tag{5.2}$$

and we show this in two steps. We first consider the case $1 \le i \le n-2$. We say that an NBC set S **admits** the bubble $\{(i,j),(\bar{i},\bar{j})\}$ when S remains an NBC set after we add the two atoms (i,j), $(\bar{i},\bar{j})$ to it. Next observe that every NBC set of D_{n-1} that admits the bubble $\{(n-1,i),(\overline{n-1},\bar{i})\}$ also admits the bubble $\{(n,i),(\bar{n},\bar{i})\}$ in D_n. Conversely every NBC set of D_n that admits the bubble $\{(n,i),(\bar{n},\bar{i})\}$ can be obtained from an NBC set of D_{n-1} that admits the bubble $\{(n-1,i),(\overline{n-1},\bar{i})\}$ by adding at most one atom from the hand in Figure 10.

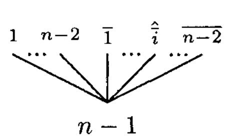


Figure 10. Added atoms to S $\in \mathrm{NBC}(D_{n-1})$.

Here $\hat{\bar{i}}$ means that it is forbidden to choose $\bar{i}$. There are $2n-5$ atoms in this hand and we have shown the recursive relation

$$P_n(t) = P_{n-1}(t)(1+(2n-5)t)$$

which proves the induction.

For the case $i = n-1$ of the NBC sets that contain the bubble $\{(n,n-1),(\bar{n},\overline{n-1})\}$, we show that from each NBC set that admits the bubble $\{(n-1,n-2),(\overline{n-1},\overline{n-2})\}$, we can construct $2n-5$ NBC sets that admit the bubble $\{(n,n-1),(\bar{n},\overline{n-1})\}$, and that each such NBC set is uniquely obtained and that we have them all.

Call A and B the collections of NBC sets of D_n that respectively admit the bubbles $\{(n-1,n-2),(\overline{n-1},\overline{n-2})\}$ and $\{(n,n-1),(\overline{n},\overline{n-1})\}$. An NBC set in A can never contain an atom of the form $(\overline{n-2},\bar{i})$ for $1 \leq i \leq n-3$.

(i) Suppose that we start with an NBC set $S \in A$ that contains no atom of the form $(n-2,i)$, $1 \leq i \leq n-3$. To obtain an element of B, we can add to S at most one atom from the set in Figure 11(a). So we have $2n-5$ choices for each of these NBC sets in A.

(ii) Suppose we now start with an NBC set $S \in A$ that contains an atom of the form $(n-2,i)$, $1 \leq i \leq n-3$. Let G_1 be the connected component of S that belongs to $(n-2,i)$ and G_2 its complement in the graph of S. Let $\{i = i_0, i_1, i_2, \ldots, i_k, n-2 = i_{k+1}\}$ be the set of vertices in G_1. We add atoms to S in three ways. First we add an atom of the form $(n-1,i_t)$, $0 \leq t \leq k+1$, to obtain a set S^+. If $S^+ \notin B$, which is possible, then we exchange the vertices $n-1$ and $n-2$ in S^+, i.e. we replace the pair of atoms $\{(n-1,i_t),(n-2,i)\}$ by the pair $\{(n-2,i_t),(n-1,i)\}$ and we obtain an NBC set of B. If the set S^+ is in B, we leave it unchanged.

Figure 11. Added atoms to $S \in A$.

The second way consists in replacing $(n-2,i)$ by $(n-1,i)$ in S and then choosing at most one atom in the hands given in Figure 11(b) where $\{j_1, j_2, \ldots, j_{n-k-4}\}$ is the set of vertices of G_2. In the third possibility, we add the atom $(\overline{n-2},\bar{i})$ to G_1 and create the bubble $\{(n-2,i),(\overline{n-2},\bar{i})\}$. Adding up these possibilities gives $(k+2)+2(n-k-4)+(k+1) = 2n-5$ choices of atoms and each NBC set of B is chosen once and only once.

Finally, the sum of the polynomial in identity (5.1) with $(n-1)$ times the polynomial in identity (5.2) gives:

$$\begin{aligned} \mathrm{NBC}(t, D_n) &= (1 + 2(n-1)t)\,\mathrm{NBC}(t, D_{n-1}) + (n-1)t^2\,\mathrm{NBC}(t, B_{n-2}) \\ &= Q_n(t) + (n-1)P_n(t) \end{aligned} \tag{5.3}$$

and hence the result. ∎

Observe that identity (5.2) suggests that there exists a bijection between the NBC sets of D_n that admit the bubble $\{(n,i),(\bar{n},\bar{i})\}$ and the elements of B_{n-2}. So far, we have been unable to establish this bijection.

ACKNOWLEDGEMENT: The problem of finding a combinatorial explanation for the factorization of $F_W(t)$ by searching for a corresponding factorization of the broken circuit complex was posed by A. Garsia in 1980. We are grateful to him for suggesting the problem to us.

While writing the final version of this paper, the authors have received a preprint ([2]) from B. Sagan and C. Bennett which intersects the contents of our paper. Their characterization of the NBC bases of D_n is similar to ours and they introduce the notion of atom-decision tree to prove the factorization of the enumerating polynomial of the NBC of D_n. We are thankful to them for letting us know about their work.

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